Probability & Statistics (1)

Conditional Probability and Independence (II)

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Independent Events

- 在前面我們討論$P(E|F)$條件機率的部分，給定F的前提下E的條件機率$P(E|F)$，通常不等於$P(E)$。

- 假設今天給定F的前提下E的條件機率$P(E|F)$ 等於$P(E)$，我們則可以說事件E與F彼此獨立，因為他們不會有相依性。

$$P(E|F) = \frac{P(EF)}{P(F)} = P(E) \Rightarrow P(EF) = P(E)P(F)$$

Independent Events

- **Definition**
- Two events E and F are said to be *independent* if $\boldsymbol{P(EF) = P(E)P(F)}$ holds.
- Two events E and F that are not independent are said to be *dependent*.

Independent Events

- 舉一個簡單的例子，假設今天投擲一顆公平的骰子，出現奇數的事件為E，而出現3的倍數事件為F。那麼出現點數為奇數下，點數為3的倍數條件機率為?

$$P(E) = \frac{1}{2}; P(F) = \frac{1}{3}; P(EF) = \frac{1}{6}$$

$$P(F|E) = \frac{P(FE)}{P(E)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3} = P(F);\ P(E|F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{6}}{\frac{1}{3}} = \frac{1}{2} = P(E)$$

Independent Events

- **範例一**

今天投擲兩個硬幣，假設這兩枚為公平的硬幣那麼就會有四種不同的結果。給定事件E為第一枚硬幣為人頭，事件F為第二枚為背面。則事件E與事件F互為獨立，因。

因為$P(EF) = P(\{(h,t)\}) = \frac{1}{4}$

因此$P(E) = P(\{((h,h),(h,t)\}) = \frac{1}{2}$，而$P(F) = (\{(h,t),(t,t)\}) = \frac{1}{2}$

Independent Events

- **範例二**

假設我們投擲兩顆公平的骰子，令事件E_1為點數合為6的事件，事件F為第一顆骰子的點數為4。則:

$$P(E_1F) = P(\{4,2\}) = \frac{1}{36}$$

$$P(E_1)P(F) = \left(\frac{5}{36}\right)\left(\frac{1}{6}\right) = \frac{5}{216}$$

$$\because P(E_1F) \neq P(E_1)P(F)$$

$$\therefore E_1 \text{ and } F \text{ are not independent}$$

Independent Events

- **範例二**

但如果今天E_2為兩個骰子點數合為7的事件，則:

$$P(E_2F) = P(\{4,3\}) = \frac{1}{36}$$

$$P(E_2)P(F) = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$$

$$\because P(E_2F) = P(E_2)P(F)$$

$$\therefore E_2 and\ F\ are\ independent$$

Independent Events

- **Proposition 1**

If E and F are independent, then so are E and F^c.

Proof:

假設E與F互為獨立。因為$E = EF \cup EF^c$與EF與EF^c很明顯的為互斥事件，所以:

$$P(E) = P(EF) + P(EF^c) = P(E)P(F) + P(EF^c)$$

或者是

$$P(EF^c) = P(E)[1 - P(F)] = P(E)P(F^c)$$

Independent Events

- Question 1
- 假設事件E與事件F互相獨立，事件E與事件G互相獨立，試問:
  事件E是否與事件FG互相獨立?

- Think about it!

Independent Events

- **範例三**

投擲兩顆公平的骰子，事件E為點數合為7，事件F為第一顆骰子點數為4，事件G為第二顆骰子的點數為3。

Solution:

$$P(E)=\frac{1}{6};P(F)=\frac{1}{6};P(G)=\frac{1}{6}$$

從此可以知道事件E與事件F為獨立事件，且事件E與事件G為獨立事件。但是事件E與事件FG呢?

$$P(E|FG)=1$$

Independent Events

- **Definition**

Three events $E, F, and\ G$ are said to be independent if

$$P(EFG) = P(E)P(F)P(G)$$

$$P(EF) = P(E)P(F)$$

$$P(EG) = P(E)P(G)$$

$$P(FG) = P(F)P(G)$$

Independent Events

Note that if $E, F, and\ G$ are independent, then E will be independent or any event formed from F and G. For instance, E is independent of $F \cup G$, since

$$P[E(F \cup G)] = P(EF \cup EG)$$
$$P[E(F \cup G)] = P(EF) + P(EG) - P(EFG)$$
$$P[E(F \cup G)] = P(E)P(F) + P(E)P(G) - P(E)P(FG)$$
$$P[E(F \cup G)] = P(E)[P(F) + P(G) - P(FG)]$$
$$P[E(F \cup G)] = P(E)P(F \cup G)$$

Independent Events

- We may extend the definition of independence to more than three events. Given a series of independent events $E_1, E_2, \ldots, E_n$, for every subset $E_{1'}, E_{2'}, \ldots, E_{r'}, r \leq n$ of these events.

$$P(E_{1'}, E_{2'}, \ldots, E_{r'}) = P(E_{1'})P(E_{2'}) \ldots P(E_{r'})$$

- We define an infinite set of events to be independent if every finite subset of those events is independent.

Independent Events

- **範例四**

假設我們進行一個無限長的臨床試驗序列，其個別試驗成功的機率為p，故失敗的機率為$1-p$。試問:

a) 在n次試驗中，至少成功一次的機率為何?
b) 在n次試驗中，在成功k次的機率為何?
c) 全部試驗成功的機率為何?

Independent Events

Solution:

a) 要計算在n次試驗中至少成功一次的機率不好算，所以我們計算這個事件的補集；意即在n次試驗中沒有任何一次成功的機率為何。另E_i為第i次試驗中失敗的事件且假設不同試驗失敗事件彼此獨立，故:

$$\because P(E_1 E_2 \cdots E_n) = P(E_1)P(E_2)\cdots P(E_n) = (1-p)^n$$

$$\therefore 1-(1-p)^n$$

Independent Events

b) 在n次試驗中成功k次，也就是失敗$n-k$次且假設不同試驗事件彼此獨立，所以機率可以直接相乘，就可以直接套用二項式訂理:

$$P\{exactly\ k\ successes\} = \binom{n}{k} p^k (1-p)^{n-k}$$

c) 全部成功的機率，就可以套(1)所有失敗事件的補集:

$$P(E_1^c E_2^c \cdots E_n^c) = p^n$$

如果套continuity property of probabilities，則可以得出:

$$P\left(\bigcap_{i=1}^{\infty} E_i^c\right) = P\left(\lim_{n\to\infty} \bigcap_{i=1}^{n} E_i^c\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} E_i^c\right) = \lim_{n} p^n = \begin{cases} o\ if\ p<1 \\ 1\ if\ p=1 \end{cases}$$

Independent Events

- **範例五**

我們如果持續投擲一對骰子進行獨立試驗，兩個點數加起來5點在兩個點數加起來7點之前出現的機率為何?

Solution:

令E_n點數和不為5點或是7點出現在第$n-1$輪，與點數和為5點出現在第n輪的事件，故機率為:

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

Independent Events

$$P\{5\ on\ any\ trial\} = \frac{4}{36}, where\ P\{7\ on\ any\ trial\} = \frac{6}{36}$$

$$P(E_n) = \left(1 - \frac{10}{36}\right)^{n-1} \frac{4}{36}$$

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{1}{9} \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} = \frac{1}{9} \times \frac{1}{1 - \frac{13}{18}} = \frac{2}{5}$$

Independent Events

令事件E為點數合為5點出現在點數合為7點之前的事件，通過第一次試驗獲得需要計算的機率: 事件F第一次點數和為5點的事件；事件G為點數和為7點的事件；事件H為第一次點數和不為5點也不是7點的事件。

$$P(E) = P(E|F)P(F) + P(E|G)P(G) + P(E|H)P(H)$$

$$P(E|F) = 1$$

$$P(E|G) = 0$$

$$P(E|H) = P(E)$$

$$P(F) = \frac{4}{36}; P(G) = \frac{6}{36}; P(H) = \frac{26}{36}$$

$$P(E) = \frac{1}{9} + P(E)\frac{13}{18} \Rightarrow P(E) = \frac{2}{5}$$

Independent Events

- **範例六**

某餐廳舉辦家庭日活動，發放n種不同的優惠卷。每獲得一張新i種優惠卷的事件彼此獨立且p_i，所有種類的機率和為$\sum_{i=1}^{n} p_i = 1$。假設你收集到k個優惠卷中，事件A_i為至少獲得一張第i種的優惠卷且$i \neq j$:

a) 求$P(A_i)$

b) 求$P(A_i \cup A_j)$

c) 求$P(A_i|A_j)$

Independent Events

Solution:

(a)

$P(A_i) = 1 - P(A_i^c) = 1 - P\{no\ coupon\ is\ type\ i\} = 1 - (1 - p_i)^k$

(b)

$P(A_i \cup A_j) = 1 - P\left(\left(A_i \cup A_j\right)^c\right)$

$= 1 - P\{no\ coupon\ is\ either\ type\ i\ or\ type\ j\} = 1 - \left(1 - p_i - p_j\right)^k$

(c)

$P(A_i | A_j) = \dfrac{P(A_i A_j)}{P(A_j)}, where\ P(A_i A_j)\ could\ be\ obtained\ from\ P(A_i \cup A_j).$

Independent Events

$$P(A_i|A_j) = \frac{P(A_iA_j)}{P(A_j)}, where\ P(A_iA_j)\ could\ be\ obtained\ from\ P(A_i \cup A_j).$$

$$P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_iA_j)$$

$$\Rightarrow P(A_iA_j) = 1 - (1-p_i)^k + 1 - (1-p_j)^k - \left[1 - (1-p_i-p_j)^k\right]$$

$$\Rightarrow P(A_iA_j) = 1 - (1-p_i)^k - (1-p_j)^k + (1-p_i-p_j)^k$$

$$\Rightarrow P(A_i|A_j) = \frac{P(A_iA_j)}{P(A_j)} = \frac{1 - (1-p_i)^k - (1-p_j)^k + (1-p_i-p_j)^k}{1-(1-p_j)^k}$$

Independent Events

- **範例七: *The Problem of the points***

獨立試驗結果成功的機率為p，失敗的機率為$1-p$。試問在m次失敗之前，有發生n次成功的機率為何? 我們可以想成今天A跟B在玩賭點數，如果A贏了可以獲得一點；反之，輸了就是B獲得一點，試問B要輸m次之前，A贏n次的機率為何?

Solution:

令$P_{n,m}$為輸m次前提下贏n次的機率，透過第一次試驗結果得出:

$$P_{n,m} = pP_{n-1,m} + (1-p)P_{n,m-1}, where\ n \geq 1, m \geq 1$$

雖然可以用邊界條件($P_{n,0}=0;\ P_{0,m}=1$)來解，但是太難計算了。。。

Independent Events

費馬認為如果要在m次失敗之前要出現至少n次成功，那麼就是至少n次成功會在$m+n-1$次的試驗中發生完畢，就即使這個試驗提早結束($< m+n-1$次)，也有必要持續進行到結束。換句話說，至少有n次成功會在全部$m+n-1$次試驗中的$m-1$次失敗後出現，也剛好會等於原來題意的n次成功必須在m次失敗之前發生。則我們可以套用範例四之(b)的解答:

$$P_{n,m} = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$

Independent Events

- **範例八**

今天有兩個人(A & B)去Las Vegas賭城旅遊，玩一個投擲硬幣的賭博遊戲。如果是正面A可以從B那邊獲得一枚代幣；反之，反面的話A必須要給B一枚代幣，一直玩到任一家破產(i.e., 沒有代幣)為止。如果每次投擲硬幣的事件皆為獨立事件，出現正面的機率為p。試問在一開始A有i枚代幣，而B有$N-i$枚代幣的狀況下，A結束遊戲的機率為何?

Independent Events

Solution:

令事件E為在一開始A有i枚代幣，而B有$N-i$枚代幣的狀況下，A用光所有代幣結束遊戲的事件，使得$P_i = P(E)$。

我們可以用第一次投擲硬幣的結果來表示$P(E)$，令事件H為第一次投擲硬幣為正面的事件:

$$P_i = P(E) = P(E|H)P(H) + P(E|H^c)P(H^c)$$
$$= pP(E|H) + (1-p)P(E|H^c)$$

結束完第一輪此時A從B那邊獲得一枚代幣，兩邊狀態更新:

A有$i+1$枚代幣，而B有$N-i-1$枚代幣

Independent Events

我們可以將其中兩個條件機率想像為$P(E|H) = P_{i+1}; P(E|H^c) = P_{i-1}$，其實也就是將起始條件更改為A有$i+1$枚代幣，而B有$N-i-1$枚代幣的概念。

此時，我們可以令$q = 1 - p$

$P_i = pP_{i+1} + qP_{i-1}, where\ i = 1,2,\ldots,N-1$

設定我們的邊界條件$P_0 = 0\ and\ P_N = 1, where\ p + q = 1$

$pP_i + qP_i = pP_{i+1} + qP_{i-1}$

$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}), i = 1,2,\ldots,N-1$

Independent Events

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}), i = 1,2,\ldots,N-1$$

從$P_0 = 0$開始。。。

$$P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p}P_1$$

$$P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$$

$$\vdots$$

$$P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1$$

$$\vdots$$

$$P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = \left(\frac{q}{p}\right)^{N-1} P_1$$

Independent Events

如果我們將前$i-1$項全部加起來，可以得到:

$$P_i - P_1 = P_1\left[\left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right]$$

$$P_i = P_1\left[1 + \left(\frac{q}{p}\right) + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right]$$

$$P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\frac{q}{p}} P_1 & if\ \dfrac{q}{p} \neq 1 \\ iP_1 & if\ \dfrac{q}{p} = 1 \end{cases}, \because P_N = 1, \Rightarrow \therefore P_1 = \begin{cases} \dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} & if\ p \neq \dfrac{1}{2} \\ \dfrac{1}{N} & if\ p = \dfrac{1}{2} \end{cases}$$

Independent Events

$$P_1 = \begin{cases} \dfrac{1-\frac{q}{p}}{1-\left(\frac{q}{p}\right)^N} & if\ p \neq \dfrac{1}{2} \\ \dfrac{1}{N} & if\ p = \dfrac{1}{2} \end{cases} \Rightarrow P_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & if\ p \neq \dfrac{1}{2} \\ \dfrac{i}{N} & if\ p = \dfrac{1}{2} \end{cases}$$

令Q_i為B贏到所有代幣的事件，且起始時A有i枚代幣，而B有$N-i$枚代幣。

$$Q_i = \begin{cases} \dfrac{1-\left(\frac{p}{q}\right)^{N-i}}{1-\left(\frac{p}{q}\right)^N} & if\ q \neq \dfrac{1}{2} \\ \dfrac{N-i}{N} & if\ q = \dfrac{1}{2} \end{cases}$$

Independent Events

當$q=\frac{1}{2}\Rightarrow p=\frac{1}{2}$，但如果$q\neq\frac{1}{2}$，

$$P_i+Q_i=\frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}+\frac{1-\left(\frac{p}{q}\right)^{N-i}}{1-\left(\frac{p}{q}\right)^N}=\frac{p^N-p^N\left(\frac{q}{p}\right)^i}{p^N-q^N}+\frac{q^N-q^N\left(\frac{p}{q}\right)^{N-i}}{q^N-p^N}$$

$$P_i+Q_i=\frac{p^N-p^{N-i}q^i-q^N+q^ip^{N-i}}{p^N-q^N}=1$$

這個結果當然與$p=q=\frac{1}{2}$相同，$\frac{i}{N}+\frac{N-i}{N}=\frac{N}{N}=1$，故:

$P_i+Q_i=1$

P(·|F) is a Probability

- **Propositions**

(a) $0 \leq P(E|F) \leq 1$

(b) $P(S|F) = 1$

(c) If $E_i, i = 1,2, \ldots, are\ mutually\ exclusive\ events, then$

$$P\left(\bigcup_{1}^{\infty} E_i | F\right) = \sum_{1}^{\infty} P(E_i|F)$$

P(·|F) is a Probability

- **Propositions**

(a) $0 \leq P(E|F) \leq 1$

Proof:

$$\because P(E|F) = \frac{P(EF)}{P(F)}, \therefore 0 \leq P(E|F) \leq 1 \Rightarrow 0 \leq \frac{P(EF)}{P(F)} \leq 1,$$

$$\because EF \subset F \Rightarrow P(EF) \leq P(F)$$

(b) $P(S|F) = 1$

Proof:

$$P(S|F) = \frac{P(SF)}{P(F)} = \frac{P(F)}{P(F)} = 1$$

P(·|F) is a Probability

(c) If $E_i, i = 1,2, \ldots, are\ mutually\ exclusive\ events, then$

$$P\left(\bigcup_{1}^{\infty} E_i | F\right) = \sum_{1}^{\infty} P(E_i|F)$$

Proof:

$$P\left(\bigcup_{i=1}^{\infty} E_i | F\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} E_i\right) F\right)}{P(F)} = \frac{P\left(\bigcup_{i=1}^{\infty} E_i\, F\right)}{P(F)}, since \left(\bigcup_{i=1}^{\infty} E_i\right) F = \bigcup_{i=1}^{\infty} E_i\, F$$

$$= \frac{\sum_{i=1}^{\infty} E_i F}{P(F)} = \sum_{i=1}^{\infty} P(E_i|F), where\ E_i E_j = \emptyset \Rightarrow E_i F E_j F = \emptyset.$$

P(·|F) is a Probability

根據前面的Proposition，如果我們令$Q(E) = P(E|F)$，則$Q(E)$可以被視為在S中的某個事件的機率函數。

因此

$$Q(E_1 \cup E_2) = Q(E_1) + Q(E_2) - Q(E_1E_2)$$

可以相等於

$$P(E_1 \cup E_2|F) = P(E_1|F) + P(E_2|F) - P(E_1E_2|F)$$

P(·|F) is a Probability

同理，我們可以定義$Q(E_1|E_2) = Q(E_1E_2)/Q(E_2)$，則:

$$Q(E_1) = Q(E_1|E_2)Q(E_2) + Q(E_1|E_2^c)Q(E_2^c)$$

$$\because Q(E_1|E_2) = \frac{Q(E_1E_2)}{Q(E_2)} = \frac{P(E_1E_2|F)}{P(E_2|F)} = \frac{\frac{P(E_1E_2F)}{P(F)}}{\frac{P(E_2F)}{P(F)}} = P(E_1|E_2F)$$

$\therefore Q(E_1) = Q(E_1|E_2)Q(E_2) + Q(E_1|E_2^c)Q(E_2^c)$就可以被表示成

$$P(E_1) = P(E_1|E_2F)P(E_2|F) + P(E_1|E_2^cF)P(E_2^c|F)$$

P(·|F) is a Probability

- **範例九**

保險公司將人群分為兩種：一種為容易發生意外，另一種很難發生意外。在任一年中，容易發生意外的人發生意外的機率為0.4，而很難發生意外的人發生意外的機率為0.2。試問新加保人在第一年發生意外的前提下，在第二年保期中發生意外的條件機率為何?

Solution:

令A為新加保人為容易發生意外的人，在第i保險年度發生意外的事件為$A_i, where\ i = 1,2$

故題目要求的是

$$P(A_2|A_1) = P(A_2|AA_1)P(A|A_1) + P(A_2|A^cA_1)P(A^c|A_1)$$

P(·|F) is a Probability

$$P(A|A_1) = \frac{P(A_1A)}{P(A_1)} = \frac{P(A_1|A)P(A)}{P(A_1)}$$

假設$P(A) = 0.3$

$$P(A_1) = P(A_1|A)P(A) + P(A_1|A^c)P(A^c)$$

$$P(A_1) = 0.4 \times 0.3 + 0.2 \times 0.7 = 0.26$$

$$P(A|A_1) = \frac{P(A_1|A)P(A)}{P(A_1)} = \frac{0.4 \times 0.3}{0.26} = \frac{6}{13}$$

$$P(A^c|A_1) = 1 - P(A|A_1) = \frac{7}{13}$$

$$\because P(A_2|AA_1) = 0.4 \; and \; P(A_2|A^cA_1) = 0.2$$

$$\therefore P(A_2|A_1) = 0.4 \times \frac{6}{13} + 0.2 \times \frac{7}{13} \approx 0.29$$

P(·|F) is a Probability

- **範例十**

今天一隻母貓生出一隻小貓，但是不知道這隻小貓的生父為何?目前已知可能是兩隻公貓的其中一隻，令生父為第一隻公貓的機率為p，則生父為第二隻公貓的機率為$1-p$。DNA基因檢測結果顯示，母貓有一組基因為(A,A)；第一隻公貓該組基因為(a,a)；第二隻公貓該組基因為(A,a)。如果小貓同一組基因為(A,a)，則生父為第一隻公貓的機率為何?

P(·|F) is a Probability

Solution:

令$M_i, where\ i = 1,2$為第i隻公貓為生父的事件；$B_{A,a}$為小貓基因為(A, a)的事件，則題目要求的是$P(M_1|B_{A,a})$

$$P\left(M_1\middle|B_{A,a}\right) = \frac{P(M_1 B_{A,a})}{P(B_{A,a})} = \frac{P\left(B_{A,a}\middle|M_1\right)P(M_1)}{P\left(B_{A,a}|M_1\right)P(M_1) + P\left(B_{A,a}\middle|M_2\right)P(M_2)}$$

$$= \frac{1 \times p}{1 \times p + (1/2) \times (1 - p)} = \frac{2p}{1 + p}$$

$$\because \frac{2p}{1 + p} > p, when\ p < 1$$

P(·|F) is a Probability

- **範例十一**

又到了做獨立試驗的時候，成功的機率為p，失敗的機率為$1-p$。試問在m次連續失敗之前，n次連續成功的機率為何?

Solution:

令事件E為在m次連續失敗之前，n次連續成功的事件。首先，我們先計算第一次試驗結果，令H為第一次試驗結果成功的事件，則:

$$P(E) = pP(E|H) + qP(E|H^c), where\ q = 1-p$$

P(·|F) is a Probability

現在已知第一次的試驗成功，那麼在m次連續失敗之前，還需要成功$n-1$次。令事件F為第二次試驗到第n次試驗結果都為成功的事件:

$$\because \textbf{\textit{Proposition}}: P(E_1|F) = P(E_1|E_2F)P(E_2|F) + P(E_1|E_2^cF)P(E_2^c|F)$$

$$\therefore P(E|H) = P(E|FH)P(F|H) + P(E|F^cH)P(F^c|H)$$

$$P(E|FH) = 1$$

如果F^cH事件發生的話，就代表第一次是結果成功，但是在第二次到第$n-1$次的試驗中會出現失敗，且會將剛剛前面所有成功的試驗作廢(跟打保齡球的全倒積分很像)。

$$\therefore P(E|F^cH) = P(E|H^c)$$

P(·|F) is a Probability

獨立試驗代表F與H之間也是彼此獨立，因為$P(F)=p^{n-1}$

故$P(E|H)=P(E|FH)P(F|H)+P(E|F^cH)P(F^c|H)$

為$P(E|H)=1\times p^{n-1}+(1-p^{n-1})P(E|H^c)$

接下來用相同的方式處理$P(E|H^c)$，令事件G為第二次試驗到第m次試驗都是失敗的事件。

$$P(E|H^c)=P(E|GH^c)P(G|H^c)+P(E|G^cH^c)P(G^c|H^c)$$

GH^c為前m次試驗都是失敗，所以$P(E|GH^c)=0$

$G^c|H^c$發生的時候，第一次試驗失敗但在接下來$m-1$次的試驗中至少有一次成功，同時這一次成功將會抹除之前所有失敗的紀錄

$$P(E|G^cH^c)=P(E|H)$$

P(·|F) is a Probability

$$\because P(G^c|H^c) = P(G^c) = 1 - q^{m-1}$$

$$\therefore P(E|H^c) = \cancel{P(E|GH^c)P(G|H^c)} + P(E|G^cH^c)P(G^c|\cancel{H^c})$$

$$\Rightarrow P(E|H^c) = (1 - q^{m-1})P(E|H)$$

此時可將上式與前面的$P(E|H) = 1 \times p^{n-1} + (1 - p^{n-1})P(E|H^c)$合併

$$P(E|H) = \frac{p^{n-1}}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}$$

$$P(E|H^c) = \frac{(1 - q^{m-1})p^{n-1}}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}$$

P(·|F) is a Probability

故得出$P(E) = pP(E|H) + qP(E|H^c)$

$$P(E) = \frac{p^n + (1-q^{m-1})qp^{n-1}}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}} = \frac{p^n + qp^{n-1} - p^{n-1}q^m}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}$$

$$P(E) = \frac{p^{n-1}(p+q) - p^{n-1}q^m}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}, where\ p + q = 1$$

$$P(E) = \frac{p^{n-1} - p^{n-1}q^m}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}} = \frac{p^{n-1}(1-q^m)}{p^{n-1} + q^{m-1} - p^{n-1}q^{m-1}}$$

P(·|F) is a Probability

- 範例十二

今天有一個箱子裡面裝有$k+1$枚硬幣，拋一次箱子之後，第i枚硬幣出現正面的機率為$i/k, where\ i=0,1,2,\ldots,k$。假設有隨機從箱子取出一枚硬幣，一直重複投擲該枚硬幣。試問在前n次投擲該枚硬幣結果都為正面，那個第$(n+1)$次投擲該枚硬幣會出現一樣的結果嗎?

Solution:

令C_i為第i枚硬幣$(i=0,1,2,\ldots,k)$被選到的事件，事件F_n為前n次投擲該枚硬幣都為正面；事件H為第$(n+1)$次為正面的事件。

P(·|F) is a Probability

本題要求的是$P(H|F_n)$

$$P(H|F_n) = \sum_{i=0}^{k} P(H|F_n C_i)P(C_i|F_n)$$

假設第i枚硬幣被選到，投擲硬幣的結果為條件獨立，因此每一次出現正面的機率為i/k

$$P(H|F_n C_i) = P(H|C_i) = \frac{i}{k}$$

$$P(C_i|F_n) = \frac{P(C_i F_n)}{P(F_n)} = \frac{P(F_n|C_i)P(C_i)}{\sum_{j=0}^{k} P\left(F_n\middle|C_j\right)P(C_j)} = \frac{(i/k)^n[1/(k+1)]}{\sum_{j=0}^{k}(j/k)^n[1/(k+1)]}$$

P(·|F) is a Probability

$$P(H|F_n) = \sum_{i=0}^{k} P(H|F_n C_i) P(C_i|F_n) = \frac{\sum_{i=0}^{k} (i/k)^{n+1}}{\sum_{j=0}^{k} (j/k)^{n}}$$

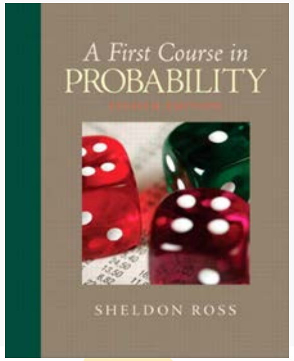
當k很大的時候，

$$\frac{1}{k}\sum_{i=0}^{k}\left(\frac{i}{k}\right)^{n+1} \approx \int_0^1 x^{n+1}dx = \frac{1}{n+2}$$

$$\frac{1}{k}\sum_{j=0}^{k}\left(\frac{j}{k}\right)^{n} \approx \int_0^1 x^{n}dx = \frac{1}{n+1}$$

故 $P(H|F_n) = \frac{n+1}{n+2}$

[#7] Assignment



- Selected Problems from Sheldon Ross Textbook [1].

3.5. An urn contains 6 white and 9 black balls. If 4 balls are to be randomly selected without replacement, what is the probability that the first 2 selected are white and the last 2 black?

3.9. Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

3.10. Three cards are randomly selected, without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade given that the second and third cards are spades.

3.19. A total of 48 percent of the women and 37 percent of the men that took a certain "quit smoking" class remained nonsmokers for at least one year after completing the class. These people then attended a success party at the end of a year. If 62 percent of the original class was male,

(a) what percentage of those attending the party were women?

(b) what percentage of the original class attended the party?

[1] Sheldon Ross. A First of Course in Probability. 8th edition.

[#7] Assignment

3.20. Fifty-two percent of the students at a certain college are females. Five percent of the students in this college are majoring in computer science. Two percent of the students are women majoring in computer science. If a student is selected at random, find the conditional probability that

(a) the student is female given that the student is majoring in computer science;

(b) this student is majoring in computer science given that the student is female.

3.33. On rainy days, Joe is late to work with probability .3; on nonrainy days, he is late with probability .1. With probability .7, it will rain tomorrow.

(a) Find the probability that Joe is early tomorrow.

(b) Given that Joe was early, what is the conditional probability that it rained?

3.43. There are 3 coins in a box. One is a two-headed coin, another is a fair coin, and the third is a biased coin that comes up heads 75 percent of the time. When one of the 3 coins is selected at random and flipped, it shows heads. What is the probability that it was the two-headed coin?

3.44. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information because he already knows that at least one of the two will go free. The jailer refuses to answer the question, pointing out that if A knew which of his fellow prisoners were to be set free, then his own probability of being executed would rise from $\frac{1}{3}$ to $\frac{1}{2}$ because he would then be one of two prisoners. What do you think of the jailer's reasoning?

Reference

Ross, S. (2010). *A first course in probability*. Pearson.

The End

If you have any questions, please do not hesitate to ask me.

Thank you for your attention))